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## Analysis of the Real Line

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# Analysis of the Real Line

by

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## Dedication

I dedicate this work to my daughter and to my son who encouraged me to “go for it” and told me that I was not too old to be a college student again.  I thank you, my biggest cheerleaders!

## Acknowledgements

I would like to thank Dr. Mark L. Daniels and Dr. Efraim Armendariz for having the confidence in me to present their course in analysis for my master's topic.

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# Abstract

## Analysis of the Real Line

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The purpose of this report is to describe the course, Analysis of the Real Line, taught at The University of Texas at Austin. Course materials are presented using the inquiry based learning method. Students work a series of warm up problems before being presented rigorous problems in calculus, including topics on integration, exponential functions, and real number line analysis. Additionally, students consider aspects of these problems that could be incorporated into a high school curriculum. Typical problems in several major areas are summarized along with warm up problems that introduce or extend the topics.

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# Chapter 1: Introduction

Since analysis is a term which addresses a wide field in mathematics, it might be best to describe Analysis of the Real Line as a course in rigorous mathematical proof. Areas of mathematics studied in this course include sets and sequences of real numbers, functions in one or more variables, and the "fundamentals" of calculus including limits, continuity, derivatives, integrals, least upper bounds and greatest upper bounds of sets, and convergence of infinite series. Analysis has been described as replacing "the equalities of calculus with inequalities: certainty with uncertainty." [**5,** p. xiii] Three major calculus theorems are studied: Fundamental Theorem of Calculus, Mean Value Theorem, and the Intermediate Value Theorem. Problems are presented and proved using proof by induction, epsilon-delta proofs, and informal proofs.

Course materials are presented by integrating topics, allowing students to review skills in calculus, trigonometry, and algebra for use later in proving theorems. For example, a rigorous proof of L'Hôpital's Rule for the $\frac{\infty}{\infty}$ case is studied early in the course. Later, students will use the rule as one method to evaluate a limit in addition to graphical techniques. In this way, students experience connections among the mathematical disciplines.

Instructors Armendariz and Daniels pose warm up problems and topical questions with very little explanation or instruction. Students work with other students, building on prior knowledge and asking questions. This method, known as *inquiry based learning*, is used to encourage students to confront and manage problem solving by active learning, or learning mathematics by doing it. With this method, the instructor acts as facilitator more than instructor. Robert L. Moore, the well known mathematician and professor of mathematics at The University of Texas from 1920 to 1969 put it this way: "*That Student is Taught the Best Who is Told the Least.*" [**4**, p. v]

The course is concluded by a day of student presentations, working in groups of two or three. Presentation assignments are taken from university level mathematics journals, and students have wide liberty in method of presentation.

# Chapter 2: The Real Number Line

## Defining the Real Numbers

Any course in Analysis of the Real Line will depend on an understanding of the properties of the set $\mathbb{R}$ of real numbers. Real numbers can be represented using a visual model of the set of all points on a number line with no spaces or gaps in between. Using a central reference point of zero, real numbers are ordered, with the negative numbers to the left and the positive numbers to the right. The unit of measure is the distance from zero to one. The computational model for real numbers uses the *infinite decimal expansion*, with an integer part followed by infinitely many decimal places.

$$\mathrm{N}.a_1a_2a_3\ldots = \mathrm{N} + \frac{a_1}{10} + \frac{a_2}{100} + \frac{a_3}{1000} + \ldots \qquad \text{where } \mathrm{N}, a_n \in \mathbb{Z} \tag{1}$$

However, there are difficulties working with these representations. For example, the number *one* can be written in two ways:

$$0.99999\ldots = 1$$

The ambiguity of these two representations is "a serious inconvenience in working theoretically with decimals." [**5**, p. 1] Furthermore, the irrational numbers require an arbitrary number of decimal places to approximate their magnitude when performing computations. From the right side of (1), clearly an understanding of sequences and limits is required. To illustrate the point, the irrational number $e$ is expressed using a converging sequence later in this chapter.

**The Completeness Axiom**

The real numbers can also be described as being complete. *Completeness* means that all numbers can be written as in (1). Additionally, when comparing the infinite number of $\mathbb{Q}$ rational numbers to the infinite number of irrational numbers, it is apparent that when numbers from 0 to 9 are chosen at random to represent $a_n$ in (1), selecting a number pattern that is either *repeating* or *terminating* (with only a string of zeros following) is far less likely than getting a pattern with no repetitions. Therefore, although an infinite number in each set exists, there are more irrationals than rationals.

The concept of completeness as used in set theory is defined as the *Completeness Property of the Real Numbers*:

If a set $S$ of real numbers has an upper bound, then $S$ has a least upper bound. [**1**]

To understand the *least upper bound*, let $S$ be a set of real numbers, with $x \in S$. Then $x$ is said to be a least upper bound if $x$ is an upper bound for $S$ and $x \le y$ for each upper bound $y$ of $S$ [**1**]. Students prove that $x$ is unique using a proof by contradiction technique: if $x_1$ and $x_2$ are both least upper bounds, by definition $x_1 \le x_2$ and $x_2 \le x_1$, so $x_1 = x_2$. Students also prove that $x$ is the least upper bound for $S$ if and only if for each $\varepsilon > 0$, there exists $a \in S$ such that $x - \varepsilon < a$.

After reviewing the definition of both open and closed intervals in the set of real numbers, the concept of the *open cover* is introduced, using a set $K$ of real numbers: "An open cover for the set $K$ is a set $\mathcal{C}$ of open intervals such that every point $x$ in $K$ belongs to some open interval in the set $\mathcal{C}$" [**1**]. Furthermore, a *finite* set of open intervals can cover any open interval ($a$, $b$) or closed interval [$a$, $b$].

As an exercise, students prove the statement:

Suppose $a, b \in \mathbb{R}$ with $a \leq b$. If a set $\mathcal{C}$ of open intervals covers [$a$, $b$], then some finite subset of open intervals from $\mathcal{C}$ covers [$a$, $b$].

First the set $S$ is shown to be nonempty. Given that the set $S$ is bounded above, the element $m$, defined as the least upper bound of $S$, is then shown to be equal to $b$.

**A Warm Up Problem Using a least upper bound**

With students gaining familiarity with least upper bounds, epsilon delta proofs, and sequences of real numbers, a theorem for the limit of a series $\{a_n\}$ is studied.

**Theorem.** If $\{a_n\}$ is a sequence of real numbers such that $a_k \leq a_{k+1}$ for $k \in \mathbb{Z}^+$ and $a_n \leq M$ for some $M \in \mathbb{R}$, then $\lim_{n\to\infty} a_n = c$, where $c = l.u.b.\{x \mid x = a_n \text{ for some } n\}$.

The definition of the limit of a sequence states that for every $\varepsilon > 0$, there is a positive integer $n$ such that $|a_k - c| < \varepsilon$ for all $k \geq n$. Since $c$ is the least upper bound (l.u.b.) of $\{a_n\}$, there exists an $n \in N$ such that

$$c - \varepsilon < a_n \leq c < c + \varepsilon.$$

For all $k \geq n$, $a_n \leq a_k \leq c$, so

$$c - \varepsilon < a_n \leq a_k < c + \varepsilon.$$

and now $|a_k - c| < \varepsilon$ for all $k \geq n$. Therefore

$$\lim_{n\to\infty} a_n = c.$$

**Another least upper bound: *e***

In the next chapters on integration and exponentiation, the irrational number $e$ will be used in examples and theorems. It is worthwhile to note that $e$ can be defined as a least upper bound of the set $E$, as follows: Let $E$ be the set of numbers

$$\left\{1, 1+\frac{1}{1!}, 1+\frac{1}{1!}+\frac{1}{2!}, 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}, 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}, \ldots\right\}$$

Then $E$ is bounded above. Its least upper bound is called $e$ and it is irrational [**2**, p.21].

# Chapter 3: A Review of Calculus

A course in Analysis of the Real Line requires a strong background in calculus. A review of calculus is introduced by using an age-old problem first chronicled by Virgil in Book 1 of the Aeneid commonly called Dido's Problem. Dido outwitted King Jarbas by maximizing land she could buy by cutting an ox hide into long thin strips and enclosing a semicircular area along the coastline, instead of using the hide's original area. [**6**, p.1] The following questions are posed [**1**]:

**The Isometric Problem.**

<u>Question 1</u>: Consider all smooth closed curves in the plane having a given length. Is there a least one which encloses a largest area?

<u>Question 2</u>: Consider all smooth closed curves in the plane all of which enclose a given area. Is there one of shortest length?

Assumptions are made that the curves in question have no crossings, the enclosed regions are convex, and the length of the curve is $2\pi$. A parametric representation describing the curve in the plane might use the relation $R(x, y) = 0$. For Question 1, the curve will be parametrized in terms of **arc length**, $s$, with $0 \le s \le 2\pi$. With $x = f(s)$ and $y = g(s)$, the curve is now described as $R(f(s), g(s)) = 0$, or, since $x$ and $y$ are functions of $s$, $R(x(s), y(s)) = 0$.

As indicated in Figure 1, the measure of the distance of $s$ begins at the point (0,0). Additionally, note that $y(\pi) = 0$.

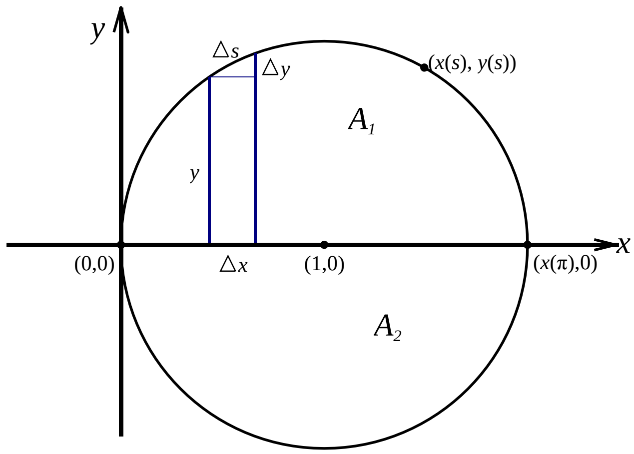


**Figure 1.** Diagram of parametrized curve, $R(x(s), y(s))$.

The curve encloses an area $A$ and is partitioned into two "halves", $A_1$ being the top portion and $A_2$, the bottom. The enclosed area can then be evaluated by integrating from 0 to π for the top region, and from π to 2π for the bottom.

A close-up of Figure 1 shows that the area of $A_1$ is an accumulation of areas of rectangles with dimensions $y$ by $\Delta x$. Because $x$ is changing with respect to $s$, the limit of $\frac{\Delta x}{\Delta s}$ will result in the derivative $\frac{dx}{ds}$, giving the areas as of the products of $y \cdot \frac{dx}{ds}$. Now $A_1$ and $A_2$ can be represented as

$$A_1 = \int_0^{\pi} y\left(\tfrac{dx}{ds}\right)ds \text{ and } A_2 = \int_{\pi}^{2\pi} y\left(\tfrac{dx}{ds}\right)ds.$$

Thus

$$A = A_1 + A_2 = \int_0^{\pi} y\left(\tfrac{dx}{ds}\right)ds + \int_{\pi}^{2\pi} y\left(\tfrac{dx}{ds}\right)ds.$$

The proof will proceed by showing that

$$A_1 \leq \frac{\pi}{2}. \tag{2}$$

If (2) is true, it follows that

$$A_2 \leq \frac{\pi}{2}$$

and

$$A_1 + A_2 \le \frac{\pi}{2} + \frac{\pi}{2} \le \pi .$$

Therefore, the maximum area would be π, the area of a circle with radius of 1, which answers Question 1.

Given the fact that the square of any real number is nonnegative, it can be shown that since

$$0 \le (a-b)^2 \text{ where } a, b \in R$$

and, by expanding the binomial and solving for *ab,* the following is true,

$$ab \le \frac{a^2 + b^2}{2}.$$

Let

$$a = y \text{ and } b = \frac{dx}{ds},$$

after substitution

$$y\left(\frac{dx}{ds}\right) \le \frac{y^2 + \left(\frac{dx}{ds}\right)^2}{2} ,$$

and integrating from 0 to π yields

$$\int_0^\pi y\left(\tfrac{dx}{ds}\right) ds \le \int_0^\pi \left[\frac{y^2 + \left(\frac{dx}{ds}\right)^2}{2}\right] ds. \tag{3}$$

However, the left side of (3) is $A_1$, so

$$A_1 \leq \frac{1}{2}\int_0^{\pi}\left[y^2 + \left(\frac{dx}{ds}\right)^2\right]ds. \tag{4}$$

A closer look at Figure 1 and the use of the Pythagorean theorem show that the arc length $s$ can be found by

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2$$

$$1 = \left(\frac{\Delta x}{\Delta s}\right)^2 + \left(\frac{\Delta y}{\Delta s}\right)^2.$$

Applying the limit,

$$1 = \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2$$

can be written

$$1 - \left(\frac{dy}{ds}\right)^2 = \left(\frac{dx}{ds}\right)^2. \tag{5}$$

Substituting (5) back into (4) gives

$$A_1 \leq \frac{1}{2}\int_0^{\pi}\left[y^2 + 1 - \left(\frac{dy}{ds}\right)^2\right]ds. \tag{6}$$

By using trigonometry and $u$-substitution, when $0 \leq s \leq \pi$,

$$y(s) = u(s)\sin(s) \text{ and } x(s) = u(s)\cos(s).$$

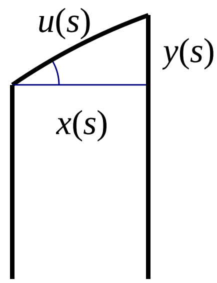


**Figure 2.** Close-up of sector $u(s)$.

Using the product rule when taking the derivative of $y(s)$,

$$\frac{dy}{ds} = \left[ u\cos(s) + \sin(s)\frac{du}{ds} \right]. \tag{7}$$

Substituting (7) into (6) yields

$$A_1 \le \tfrac{1}{2}\int_0^{\pi} \left[ (u(s)\sin(s))^2 + 1 - \left( u(s)\cos(s) + \sin(s)\frac{du}{ds} \right)^2 \right] ds. \tag{8}$$

After algebraic manipulation and the use of trigonometric double angle identities, (8) can now be expressed as

$$A_1 \le \tfrac{1}{2}\int_0^{\pi} \left[ -u^2\cos(2s) - u(s)\sin(2s)\frac{du}{ds} - \left(\frac{du}{ds}\right)^2 \sin^2(s) + 1 \right] ds. \tag{9}$$

In order to evaluate the integral, it is necessary to use integration by parts. To begin, in the first expression of the integrand, using $\int kdv = kv - \int vdk$ as the model for integration by parts, let $k = u^2$ and $v = \tfrac{1}{2}\sin 2s$. Thus

$$\int_0^{\pi} \left[ u^2\cos(2s) \right] ds = \tfrac{1}{2}u^2\sin(2s) - [-\tfrac{1}{2}u\cos(2s) + \int_0^{\pi} \tfrac{1}{2}\cos(2s)]. \tag{10}$$

Let $k = u$ and $v = -\tfrac{1}{2}\cos 2s$, integrate (10) by parts a second time and evaluate. Thus

$$\int_0^{\pi}\left[u^2\cos(2s)\right]ds=\left[\tfrac{1}{2}u^2\sin(2s)+\tfrac{1}{2}u\cos(2s)-\tfrac{1}{4}\sin(2s)\right]\Big|_0^{\pi}$$

$$=\tfrac{1}{2}u-\tfrac{1}{2}u=0.$$

Similarly the second expression in (9) is evaluated, producing a value of zero. However, the last two expressions of the integrand will provide the requisite part for the assumption in (2),

$$A_1\le\tfrac{1}{2}\int_0^{\pi}\left[1-\left(\frac{du}{ds}\right)^2\sin^2(s)\right]ds. \tag{11}$$

Because both $\left(\frac{du}{ds}\right)^2\ge 0$ and $\sin^2(s)\ge 0$, then $\left(\frac{du}{ds}\right)^2\sin^2(s)\ge 0$.

Now,

$$1-\left(\frac{du}{ds}\right)^2\sin^2(s)\le 1.$$

Integrating both sides of the inequality

$$A_1\le\tfrac{1}{2}\int_0^{\pi}\left[1-\left(\frac{du}{ds}\right)^2\sin^2(s)\right]ds\le\tfrac{1}{2}\int_0^{\pi}1ds$$

and evaluating the definite integral yields

$$A_1\le\tfrac{1}{2}\int_0^{\pi}\left[1-\left(\frac{du}{ds}\right)^2\sin^2(s)\right]ds\le\frac{\pi}{2}.$$

By the squeeze theorem

$$A_1\le\frac{\pi}{2}.$$

The conjecture requiring (2) from Question 1 confirms that the greatest area is a circle, with area equal to pi.

Question 2 is a corollary of Question 1. Indeed Dido had the right idea!

# Chapter 4: Another Look at Integration

## A Number Between 2 and 3

The irrational number *e* plays a major role in many aspects of real analysis. With a simple construction, *e* can be portrayed graphically in relation to the area under the curve of the function

$$f(t) = \frac{1}{t}. \tag{12}$$

Students construct the graph of the function on grid paper as shown in Figure 3.

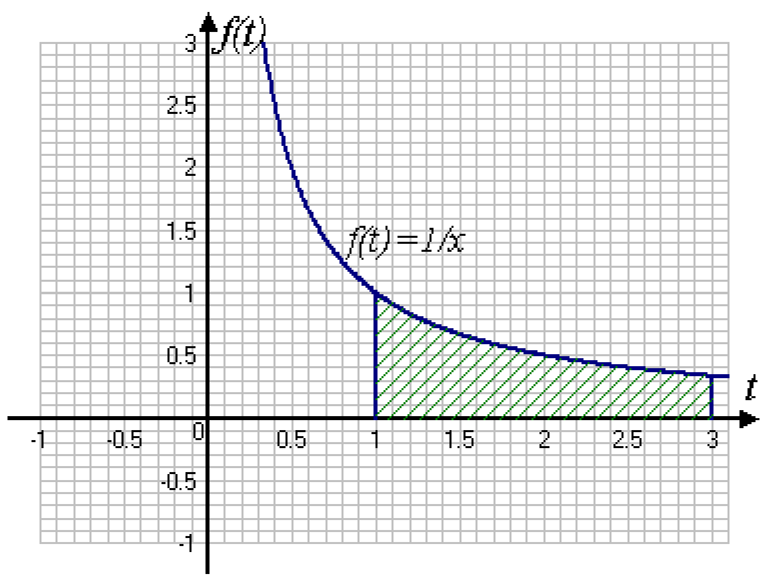


**Figure 3.** Accumulation graph of the function $L(x)$ from 1 to 3.

$L(x)$ is the function that represents the accumulated area under $f(t)$ on the interval $[1, x]$ where $x \in [1,3]$. From the exploration, students find these values: $L(1) = 0$, $L(2) = 0.68$ and $L(3) = 1.075$. The Intermediate Value Theorem guarantees that there is a unique value $x \in [1,3]$ such that $L(x) = 1$. This value of $x$ is equal to $e$ which is approximately 2.71828. Thus,

$$\int_1^e \frac{1}{x}\,dx = 1.$$

**Warm Up on a Bounded Region**

The following warm up problem ties together an area calculation of a bounded region, $A(x)$, with a calculation of $A'(x)$, leading students to an exploration of the Fundamental Theorem of Integral Calculus.

From Figure 4, students (a) use geometry and trigonometry to find an algebraic expression for $A(x)$ and (b) calculate $A'(x)$.

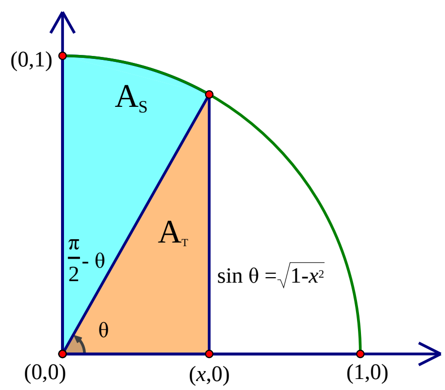


**Figure 4.** $A(x)$ = the measure of the region bounded by $f(t) = \sqrt{1-t^2}$ on $[0, x]$.

For part (a), Figure 4 shows bounded region $A(x)$ divided by using the radius of length one into two areas creating a triangle ($A_T$) and a sector ($A_S$). The two areas can be represented as

$$A_T = \frac{1}{2}x(\sin\theta)$$

and

$$A_S = \frac{\frac{\pi}{2}-\theta}{2\pi}(\pi) \text{ or } A_S = \frac{1}{2}\left(\frac{\pi}{2}-\theta\right).$$

Thus,

$$A(x) = \frac{1}{2}\left(x\cdot\sin\theta + \frac{\pi}{2} - \theta\right). \tag{13}$$

Substituting $\sin\theta = \sqrt{1-x^2}$ and $\theta = \cos^{-1} x$ in (13),

$$A(x) = \frac{1}{2}\left(x\sqrt{1-x^2} + \frac{\pi}{2} - \cos^{-1} x\right).$$

Now for part (b) differentiating $A(x)$ produces

$$A'(x) = \frac{1}{2}\left[x\cdot\frac{1}{2}(1-x^2)^{-\frac{1}{2}}(2x) + \sqrt{1-x^2} + 0 + (1-x^2)^{-\frac{1}{2}}\right].$$

After algebraic manipulation,

$$A'(x) = \frac{1}{2}\left(2\sqrt{1-x^2}\right).$$

Therefore,

$$A'(x) = \sqrt{1-x^2}.$$

The warm up has effectively shown how integration and differentiation are related. The relationship was originally described independently by Newton and Liebniz who are attributed with the invention of the calculus: “both men realized that the basic processes of finding tangents and areas, that is, differentiating and integrating, are mutually inverse—what we now call the Fundamental Theorem of Calculus [**3**, p. 186]."

**The Fundamental Theorem of Integral Calculus**

From these two problems, students have informally been introduced to the Fundamental Theorem of Integral Calculus. In the bounded region exercise, a cumulative area function $A(x)$, associated with a continuous function $f(x)$ over an interval is used to demonstrate that the derivative of $A(x)$ is equal to the original function $f(x)$.

The theorem is then formally explored by defining the function $y = f(t),\ f(t) \geq 0$ for all $t \in [a,b]$ and where

$$A(x) = \int_a^x f(t)dt. \tag{14}$$

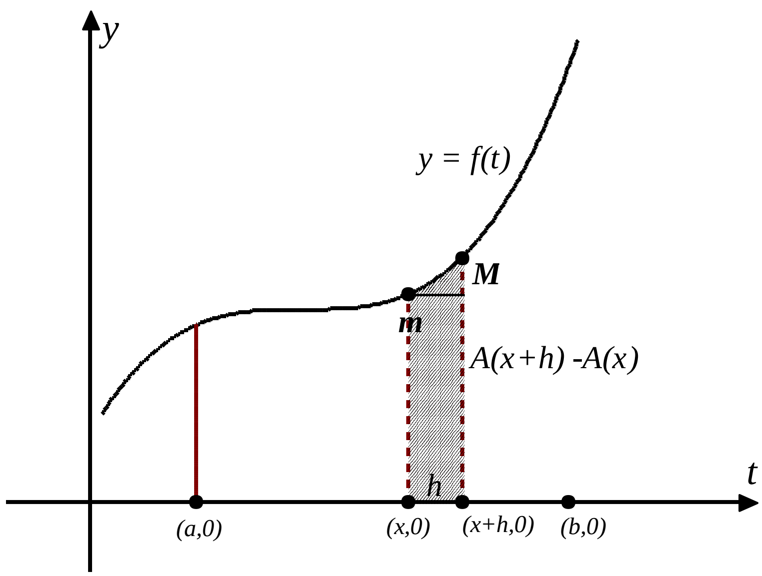


**Figure 5.** Graph of $A(x) = \int_a^x f(t)dt$.

From Figure 5, since $f$ is continuous on $[a,b]$, $f$ is continuous on $[x, x+h]$, so it has a maximum value, $M_h$ and a minimum value, $m_h$. By examining the rectangle formed, it can be seen that

$$h \cdot m_h \leq A(x+h) - A(x) \leq h \cdot M_h$$

and

$$m_h \leq \frac{A(x+h) - A(x)}{h} \leq M_h . \tag{15}$$

Taking the right hand limit of (15),

$$\lim_{h \to 0^+} m_h \leq \lim_{h \to 0^+} \frac{A(x+h) - A(x)}{h} \leq \lim_{h \to 0^+} M_h$$

$$f(x) \leq \lim_{h \to 0^+} \frac{A(x+h) - A(x)}{h} \leq f(x)$$

$$A'(x) = \lim_{h \to 0^+} \frac{A(x+h) - A(x)}{h} = f(x)$$

$$A'(x) = f(x).$$

Repeating for the left hand limit, it has been demonstrated that $A(x)$ is an antiderivative of $f(x)$, or

$$\frac{d}{dx} \int_a^x f(t)dt = f(x).$$

For the second part of the fundamental theorem, if $G$ is any function such that $G'(x) = f(x)$ for all $x$ in $[a,b]$, students show that

$$\int_a^b f(t)dt = G(b) - G(a)$$

Students have already explored the fact that if $A'(x) = G'(x)$ then $G(x) = A(x) + c$. Since

$$G(a) = A(a) + c \text{ and } G(b) = A(b) + c,$$

then

$$G(b) - G(a) = \left[A(b) + c\right] - \left[A(a) + c\right]$$

$$G(b) - G(a) = A(b) + c - A(a) - c$$

$$G(b) - G(a) = A(b) - A(a).$$

However, $A(a) = 0$. Thus,

$$G(b) - G(a) = A(b)$$

From (14),

$$G(b) - G(a) = \int_a^b f(t)dt.$$

The relationship between integration and differentiation as pioneered by Newton and Liebniz can now be appreciated through the use of analysis. The Fundamental Theorem of Integral Calculus allows the computation of definite integrals using antiderivatives and the use of definite integrals to construct antiderivatives.

# Chapter 5: The Exponential Function $y = e^x$

An exploration of the relationship between the exponential function $y = e^x$ and its inverse, $y = \ln(x)$, provides students another look at the irrational number *e*. In chapter 4, it was shown that in order for the area under the curve of the function defined as $f(t) = \frac{1}{t}$ to be equal to 1, the interval of accumulation was established to be [1, *e*].

$$y = \int_1^e \frac{1}{t}\,dt = 1$$

By defining

$$\ln(x) = \int_1^x \frac{1}{t}\,dt = 1$$

where x > 0, then $\ln(x)$ has the properties:

$$\ln(1) = 0$$

$$\ln(x) > 0 \text{ if } x > 1$$

$$\ln(x) < 0 \text{ if } 0 < x < 1.$$

Using the Fundamental Theorem of Integral Calculus,

$$\frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x}. \qquad (16)$$

Students have already seen from earlier area constructions that

$$\ln(3) = \int_1^3 \frac{1}{t}\,dt > 1$$

and

$$\ln(2) = \int_1^2 \frac{1}{t}\,dt < 1.$$

By the Mean Value Theorem, the function must have a value that makes $\ln(x) = 1$, and that value is *e*, or

$$\ln(e) = \int_1^e \frac{1}{t}\,dt = 1.$$

The function $\ln(x)$ is called the *natural logarithm* function because it "acts like a logarithm [**1**]." Students are given convincing evidence through proofs that

(a) $\ln(ab) = \ln(a) + \ln(b)$, and

(b) $\ln(x^p) = p \cdot \ln(x)$.

From part (b) it is seen that $y = e^x$ is the inverse function for $y = \ln(x)$.

$$\ln(e^p) = p \cdot \ln(e) = p \tag{17}$$

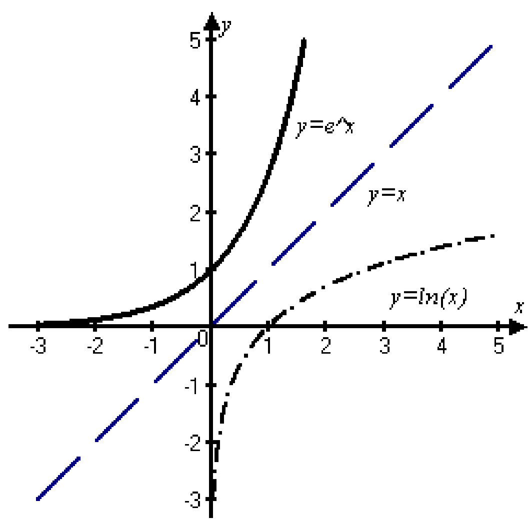


**Figure 6.** The graphs of $y = e^x$, $y = x$, and $y = \ln(x)$ [**1**]

From Figure 6, $\ln(x)$ is a 1-1 function because $\ln(x)$ is strictly increasing on $(0, \infty)$.

$$\ln(x) = \ln(x) \cdot 1$$

$$\ln(x) = \ln(x) \cdot \ln(e).$$

By (17)

$$\ln(x) \cdot \ln(e) = \ln\left(e^{\ln(x)}\right),$$

so now

$$\ln(x) = \ln\left(e^{\ln(x)}\right),$$

and

$$x = e^{\ln(x)}. \tag{18}$$

Defining $f(x) = e^x$ and $g(x) = \ln(x)$, it can be shown from (17) and (18) that $(f \circ g)(x) = (g \circ f)(x) = x$ and therefore the functions are inverses of each other [**1**].

In a subsequent exploration, students find the derivative of $e^x$ by first taking the derivative of both sides of the equation $x = \ln(e^x)$, applying the chain rule, and then using (16) to see that

$$1 = \frac{1}{e^x} \cdot \frac{d}{dx} e^x.$$

Multiplying both sides by $e^x$,

$$e^x = \frac{d}{dx} e^x.$$

Therefore, the derivative of $e^x$ is $e^x$.

## Chapter 7: Conclusion

Instructors Daniels and Armendariz have taken a carefully selected group of problems from several areas of study in mathematics to challenge students to think analytically. A strong background in calculus and differential equations is assumed for understanding the mechanics of the problems addressed in the course. However, by skillfully imbedding the use of higher algebra skills, proofs by induction, graphing, and general problem solving, this course is made accessible to students returning to the university setting with varying experience in teaching in the secondary classroom.

The explorations of the irrational number $e$ throughout this course are useful for high school teachers when introducing second year algebra students to the completeness of the real number system. With handheld calculators students can explore the tables and graph of functional representations of $e$ to discern a pattern of values and a graph that approaches a certain number, $e$, giving students an idea of the concept of a limit. The function can be revisited in both Algebra and Precalculus courses when studying continuous compounding interest.

Throughout the course inverse functions are used to build on related concepts. For example, when comparing the exponential function, with a base of $e$, to the natural logarithmic function students can picture the inverse relationship. The exploration using the area under the reciprocal function to arrive at $e$ can be introduced as early as Geometry, by estimating areas under curves, and then again in Algebra when discussing $e$ and its significance.

Students in Algebra classes can learn the rudiments of epsilon delta proofs if the definition of the absolute value function is introduced using tolerance problems. Graphical mappings such as those used throughout this course can also help Algebra and

Precalculus students see the connection between small changes in the input related to small changes in the output. Absolute value properties, usually introduced in the second course in Algebra, can be enriched with proofs, including the proof of the triangle inequality property seen in this course.

Finding sums of finite and infinite series, useful in teaching concepts of limits in calculus, can also be introduced in Algebra and Precalculus classes as a way to familiarize students with the concepts of convergence and divergence. The derivation of the formula for the sum of a geometric series presented in this course provides a good exercise for an Algebra or Precalculus student, affording a deeper understanding than just memorization of a formula.

Described in this report are just a few of the mathematical explorations presented in this course on real analysis. In addition to higher level thinking problems and projects in calculus and analysis, students experienced the inquiry based nature of the Moore method of learning which can be effective in a high school setting as well. The connections made by integrating and spiraling multiple disciplines within mathematics help students develop deeper understanding of concepts. And lastly, by incorporating the historical references of the development of analysis as well as other topics in mathematics, continuity and relevance can be brought into the classroom. Students of all ages can appreciate that from the minds of curious men and women over a period of hundreds of years have come the answers to complex problems still relevant in today's world.

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## Vita

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